Coordinated Market Mechanism for Economic Dispatch in Active Distribution Grids (Preprint)

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***Abstract*—This paper proposes a coordinated market mechanism for implementing an economic dispatch problem for active distribution grids. It is assumed that Distributed Generators (DGs) are operating in the distribution grid and offer dispatchable active and reactive power to the Distribution System Operator (DSO). The DSO economically dispatches its local DGs and transmission grid supply, while managing distribution grid voltages, losses and congestion. The main features of the proposed iterative method is that it i) deploys adequate power flow representation, ii) preserves the respective sensitive information of DGs/DSO and iii) maintains computational inexpensiveness at each iteration. The method is tested with two scenarios on 33-bus system, where it is shown to converge to the same solution as the central economic dispatch problem.**

***Index Terms*—Active Distribution Grids, Economic Dispatch, Market Mechanism, Distributed Generators**

I. INTRODUCTION

In pursuit of modernizing the electric power industry, distribution grids are subjected to new technologies (such as distributed generators (DGs)), requiring new operation strategies. This paves the way for distribution grids, traditionally a passive component of power systems, to become Active Distribution Grids (ADGs). Even though actions to be taken by ADGs are part of ongoing research [1], it is becoming quite clear that the future Distribution System Operator (DSO) is going to take up many more responsibilities.

One recently pointed task for the DSO originates in the form of trading energy among its local DGs and transmission grid interconnection [2]. In transmission grids, this task is automated through an Economic Dispatch (ED) problem [3], where generators/elastic demand submit their preferences and a central entity maximizes the overall social welfare. However, the ED problem from the transmission grid may not be simply transfered to distribution grids. This is because, as compared to transmission grids, distribution grids; i) have higher nonlinearity in their power flows; ii) are radial in nature; and iii) have privacy sensitive entities (e.g. private DGs).

To this end, there exists some direct and indirect[1] works towards the distribution grid ED problem. From iterative/distributed solution techniques, authors in [4]–[7] deploy simplified (DC) power flows, whereas the works in [8]–[10] include convexified power flow formulation. Using approximated (linear) power flow formulation, works in [11]–[13] demonstrate that the DSO is able to clear intuitive market products such as energy, loss and congestion (line flow and voltage) components.

Authors in [4]–[7] consider privacy among market participants, however, using an inadequate DC power flow for representing distribution grids. Adequate power flows are important in distribution grid market formulation as they are central to the individual dispatch decisions problem [8], [9], [11], [13]. The distributed convexified solution algorithms [8]–[10] appoint individual buses (DGs) to monitor their local grid quantities and exchange information with their neighboring buses. This approach may suffer from regulatory issues, as the DSO is usually responsible for handling grid quantities and might not make this information public to DGs. Though computationally attractive, the centralized market clearing of [11]–[13] suffer from the fact that the monetary driven (DGs) might not feel comfortable sharing their energy requirements and costs function with the central entity (DSO).

This paper addresses the above mentioned issues and provides three contribution to the state-of-the-art literature on distribution grid ED.

1) For constructing convex ED problem, we utilize fixed-point distribution grid *load-flow* form [14] for deriving approximate (linear) power flows. This approximation is not only computationally inexpensive [15], but can also be extended for the case of multi-phase and unbalanced grids [16], [17].
2) While representing grid conditions adequately, we propose a computationally inexpensive iterative coordinated market mechanism, in which each entity is only required to perform simple arithmetics (vector matrix multiplications).
3) The proposed method respects privacy constraints of both the DSO and DGs. This means that DGs get to keep their local energy requirements and cost functions whereas only the DSO handles grid quantities.

The rest of the paper is organize as follows. Section II describes the system model. Section III explains the proposed method, with simulation results presented in Section IV and Section V concluding the paper.

Notations: scalars are lower case x; vectors are bold and lower case $(\mathbf{x})$; matrices are bold and upper case $(\mathbf{X})$. For complex $(\mathbf{Y})$ real part is $\Re(\mathbf{Y})$, imaginary part is $\Im(\mathbf{Y})$ and complex conjugate is $\overline{\mathbf{Y}}$. Matrix $(\mathbf{X})_\mathrm{n}$ correspondingly selects first n rows of $\mathbf{X}$, and $(\mathbf{X})_{i,j}$ selects entry (i,j) from $\mathbf{X}$; $\mathrm{diag}(\mathbf{x})$ turns vector $\mathbf{x}$ to a matrix with $\mathbf{x}$ at its diagonal; $\mathbf{1}_\mathrm{n}$ is the vector of size n with all entries as 1; the approximated and actual value of $\mathbf{x}$ is given by $\tilde{\mathbf{x}}$ and $\hat{\mathbf{x}}$, respectively.

This work is supported by the National Research Foundation (NRF) Singapore under its Campus for Research Excellence And Technological Enterprise (CREATE) program.

[1]We consider demand response strategies, optimal flexible resource allocation in distribution grids as an indirect form of the distribution grid ED problem.

II. System Model

A. Grid Model

We assume a balanced portion of grid with steady state at bus i described by; complex voltage $u_i = v_i e^{j\theta_i} \in \mathbb{C}$, with magnitude $v_i \in \mathbb{R}$ and angle $\theta_i \in \mathbb{R}$; and complex injections $s_i = p_i + jq_i$, where $p_i \in \mathbb{R}$ and $q_i \in \mathbb{R}$ are the active and reactive powers. For the whole grid, let set of buses be indexed as: $\{0, 1, \ldots, \mathrm{n}\} \in \mathrm{N}$, where 0 is for the root-bus and the rest are n constant active reactive injection (PQ) buses, i.e., active and reactive powers are imposed independently at buses without considering the effect of voltages [16]. For the whole grid, we have; the complex voltage vector $\mathbf{u} := (u_0, (\mathbf{u}_\mathrm{L})^\intercal)^\intercal \in \mathbb{C}^\mathrm{N}$, where $\mathbf{u}_\mathrm{L} := (u_1, \ldots, u_\mathrm{n})^\intercal \in \mathbb{C}^\mathrm{n}$ and $u_0 \in \mathbb{C}$ are the voltages for n buses and the root-bus, respectively; complex power vector $\mathbf{s} := (s_0, (\mathbf{s}_\mathrm{L})^\intercal)^\intercal \in \mathbb{C}^\mathrm{N}$, where $s_0 \in \mathbb{C}$ and $\mathbf{s}_\mathrm{L} := \mathbf{p}_\mathrm{L} + j\mathbf{q}_\mathrm{L}$. The active and reactive power injections $\mathbf{p}_\mathrm{L}/\mathbf{q}_\mathrm{L} := (\mathbf{p}^\mathrm{dg} - \mathbf{p}^\mathrm{cl})/(\mathbf{q}^\mathrm{dg} - \mathbf{q}^\mathrm{cl})$ contain both injections from DGs $\mathbf{p}^\mathrm{dg}/\mathbf{q}^\mathrm{dg} := (p_1^\mathrm{dg}/q_1^\mathrm{dg}, \ldots, p_n^\mathrm{dg}/q_n^\mathrm{dg})^\intercal$ and fixed (constant) loads $\mathbf{p}^\mathrm{cl}/\mathbf{q}^\mathrm{cl} := (p_1^\mathrm{cl}/q_1^\mathrm{cl}, \ldots, p_n^\mathrm{cl}/q_n^\mathrm{cl})^\intercal$.

Remark 1. See [16, Remark 2] for more information regarding the assumption of load and DG as a PQ type injection bus. Interested reader may also refer to [18] to incorporate more general (ZIP) injection models.

Remark 2. The number of DGs are kept as n for brevity. The extension to account for arbitrary number of DGs is straightforward and is also implemented in Section IV of this paper.

B. Load-flow Problem

With the constant PQ buses, we have the following voltage to power relationship in the grid

$$\mathbf{s} = \mathrm{diag}(\mathbf{u})\overline{\mathbf{Y}}\overline{\mathbf{u}}. \tag{1a}$$

The distribution grid *load-flow* problem is then to obtain complex voltages $\mathbf{u}_\mathrm{L}$ for n PQ buses, given the specified loading $\mathbf{s}_\mathrm{L}$ and the desired voltage at the root-bus ${u_0}^2$. To this end, first, we split (1) between root-bus and n buses,

$$\mathbf{s}_\mathrm{L} = \mathrm{diag}(\mathbf{u}_\mathrm{L})(\overline{\mathbf{Y}}_{\mathrm{L}0}\overline{u}_0 + \overline{\mathbf{Y}}_\mathrm{LL}\overline{\mathbf{u}}_\mathrm{L}). \tag{2}$$

where the nodal admittance matrix $\mathbf{Y} \in \mathbb{C}^{(\mathrm{n}+1)\times(\mathrm{n}+1)}$ is decomposed as $\mathbf{Y} = \begin{bmatrix}\mathbf{Y}_{00} & \mathbf{Y}_{0\mathrm{L}} \\ \mathbf{Y}_{\mathrm{L}0} & \mathbf{Y}_\mathrm{LL}\end{bmatrix}$ where $\mathbf{Y}_{00} \in \mathbb{C}^1$, $\mathbf{Y}_{0\mathrm{L}} \in \mathbb{C}^{1\times\mathrm{n}}$, $\mathbf{Y}_{\mathrm{L}0} \in \mathbb{C}^{\mathrm{n}\times 1}$ and $\mathbf{Y}_\mathrm{LL} \in \mathbb{C}^{\mathrm{n}\times\mathrm{n}}$. Now solving for $\mathbf{u}_\mathrm{L}$ gives the following fixed-point form [14]:

$$\mathbf{u}_\mathrm{L} = \mathbf{w} + {\mathbf{Y}_\mathrm{LL}}^{-1}\mathrm{diag}(\overline{\mathbf{u}}_\mathrm{L})^{-1}\overline{\mathbf{s}}_\mathrm{L}, \tag{3}$$

where $\mathbf{w} := -{\mathbf{Y}_\mathrm{LL}}^{-1}\mathbf{Y}_{\mathrm{L}0}u_0$ is the no-load voltage. Equation (3) can be solved iteratively, for each iteration k as:

$$\mathbf{u}_\mathrm{L}^{k+1} = \mathbf{G}_{u^0,\mathbf{s}_\mathrm{L}}(\mathbf{u}_\mathrm{L}^k), \tag{4}$$

where $\mathbf{u}_\mathrm{L}^k$ is voltage at iteration k, $\mathbf{s}_\mathrm{L}$ is the loading (known injections), u^0 is the imposed root-bus voltage and $\mathbf{G}_{u^0,\mathbf{s}_\mathrm{L}}(\cdot)$ defined in (3).

Remark 3. Three advantages exists for deploying the fixed-point load-flow form (4). First, [14] provides computationally inexpensive conditions to check for solution uniqueness and existence of (4). Second, simple arithmetic operations exists to obtain linear approximation of voltages from (3) [15]. Third, the load-flow formulation along with price calculations are easily extensible to multi-phase and unbalanced grids [16], [17].

Let $\hat{\mathbf{u}}_\mathrm{L}$ be the solution upon convergence of (4). Now relevant for the ED problem, we derive expressions for line currents and system losses. Consider bus i and j connected by a distribution line element (i, j), then the complex line current i_{ij} follows

$$i_{ij} = y_{ij}^{sh}\hat{u}_i + y_{ij}^l(\hat{u}_i - \hat{u}_j) \tag{5}$$

where y_{ij}^{sh} and y_{ij}^l are the positive sequence standard distribution line's π-model shunt and line admittance [19]. Similarly, complex losses s_{ij}^l for line (i, j) is,

$$s_{ij}^\mathrm{l} = z_{ij}^l i_{ij}\overline{i_{ij}} \tag{6}$$

where line impedance $z_{ij}^l = (y_{ij}^l)^{-1}$. From (6), real power and reactive power losses respectively are $p_{ij}^\mathrm{l} := \Re(s_{ij}^\mathrm{l})$ $q_{ij}^\mathrm{l} := \Im(s_{ij}^\mathrm{l})$. For the whole grid, with m lines, we then have

$$\mathbf{i}_{ij} = \mathbf{Y}_0^\mathrm{f}u_0 + \mathbf{Y}_\mathrm{L}^\mathrm{f}\hat{\mathbf{u}}_\mathrm{L}, \tag{7}$$

$$s^\mathrm{l} = \mathbf{1}_\mathrm{m}^\intercal\mathbf{Z}^l\,\mathrm{diag}(\mathbf{i}_{ij})\overline{\mathbf{i}_{ij}}, \tag{8}$$

where $\mathbf{Z}^l := \mathrm{diag}(z_{ij}^l) \in \mathbb{C}^{\mathrm{m}\times\mathrm{m}}$, $\mathbf{Y}^\mathrm{f} := (\mathbf{Y}_0^\mathrm{f}, \mathbf{Y}_\mathrm{L}^\mathrm{f}) \in \mathbb{C}^{\mathrm{m}\times(\mathrm{n}+1)}$ arranges y_{ij}^{sh} and y_{ij}^l in an appropriate form. Note that (7) complements the distribution grid setting of input u_0 and desired output $\hat{\mathbf{u}}_\mathrm{L}$.

C. Social Welfare Model

ADG is assumed to contains DGs, optimizing their instantaneous active and reactive powers independently, with linear constraints on their maximum minimum generation capabilities [13]. Also, it is assumed that DGs are price-taking, utility maximizing agents. To this end, the overall social welfare $w_p(\mathbf{p}^\mathrm{dg})$ as the aggregate benefit of DGs from the active power procurement is taken as:

$$w_p(\mathbf{p}^\mathrm{dg}) = -\mathbf{c}^\mathrm{dg}(\mathbf{p}^\mathrm{dg}), \tag{9}$$

which is simply the negative cost of supplying the energy $\mathbf{c}^\mathrm{dg}(\mathbf{p}^\mathrm{dg})$, interpret-able as follows: Let λ be the market cleared price DGs experienced, then DGs intends to maximize the following:

$$\mathbf{p}^\mathrm{dg} = \max\ \lambda\mathbf{p}^\mathrm{dg} - \mathbf{c}^\mathrm{dg}(\mathbf{p}^\mathrm{dg}) = \max\left\{0, \left\{\mathbf{p}^\mathrm{dg} \mid \frac{\partial\mathbf{c}^\mathrm{dg}(\mathbf{p}^\mathrm{dg})}{\partial\mathbf{p}^\mathrm{dg}} = \lambda\right\}\right\}.$$

Definition 1. We define cost function $\mathbf{c}^\mathrm{dg}(\mathbf{p}^\mathrm{dg}) := \mathbf{a}_\mathbf{p}^\intercal\mathbf{p}^\mathrm{dg} + 0.5{\mathbf{p}^\mathrm{dg}}^\intercal\mathbf{B}_\mathbf{p}\mathbf{p}^\mathrm{dg}$ where $\mathbf{a}_\mathbf{p} \in \mathbb{R}^\mathrm{n}$ is a positive price per unit vector (in \$/MWh) and $\mathbf{B}_\mathbf{p} \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ in \$/MWh^2 is a symmetric, positive definite matrix simply made by placing small positive coefficients at all diagonals ($(\mathbf{B}_\mathbf{p})_{i,i}, \forall i \in \mathrm{n}$).

Remark 4. From Definition 1, $\mathbf{c}^\mathrm{dg}(\mathbf{p}^\mathrm{dg})$ in (9) is strictly convex.

As reactive power pricing has been the subject of recent interest [9], we also consider social welfare of reactive power procurement from DGs as:

$$w_q(\mathbf{q}^\mathrm{dg}) = -\mathbf{c}^\mathrm{dg}(\mathbf{q}^\mathrm{dg}), \tag{10}$$

In the end, the cost of active/reactive power supplied to the distribution grid through bulk system is modeled as $c(p_0)/c(q_0)$.

[2] Usually, voltage magnitude v_0 is aimed to be kept at 1–1.05 per unit and angle θ_0 as a reference at 0 degrees.

Remark 4 is also considered true for the reactive powers $\mathbf{q}^{\rm dg}$ and bulk power supply p_0/q_0.

D. Central Distribution Grid ED

The central distribution grid ED problem follows:

$$\begin{aligned}
\underset{\mathbf{p}^{\rm dg},\mathbf{q}^{\rm dg},p_0,q_0}{\text{maximize}}\quad & w_p(\mathbf{p}^{\rm dg})+w_q(\mathbf{q}^{\rm dg})-c(p_0)-c(q_0) && \text{(11a)}\\
\text{subject to}\quad & p_0+\mathbf{1}_{\rm n}^\top\mathbf{p}_{\rm L}=\tilde{p}^{\rm l} \quad &: \lambda_p & \text{(11b)}\\
& q_0+\mathbf{1}_{\rm n}^\top\mathbf{q}_{\rm L}=\tilde{q}^{\rm l} \quad &: \lambda_q & \text{(11c)}\\
& |\tilde{\mathbf{i}}_{ij}|\le|\mathbf{i}_{ij}^+| \quad &: \boldsymbol{\mu}^+_{|\mathbf{i}_{ij}|} & \text{(11d)}\\
& \mathbf{v}_{\rm L}^-\le\tilde{\mathbf{v}}_{\rm L}\le\mathbf{v}_{\rm L}^+ \quad &: \boldsymbol{\mu}^-_{\mathbf{v}_{\rm L}},\boldsymbol{\mu}^+_{\mathbf{v}_{\rm L}} & \text{(11e)}\\
& \mathbf{p}^{\rm dg-}\le\mathbf{p}^{\rm dg}\le\mathbf{p}^{\rm dg+} \quad &: \boldsymbol{\mu}^{\rm dg-}_{\mathbf{p}},\boldsymbol{\mu}^{\rm dg+}_{\mathbf{p}} & \text{(11f)}\\
& \mathbf{q}^{\rm dg-}\le\mathbf{q}^{\rm dg}\le\mathbf{q}^{\rm dg+} \quad &: \boldsymbol{\mu}^{\rm dg-}_{\mathbf{q}},\boldsymbol{\mu}^{\rm dg+}_{\mathbf{q}} & \text{(11g)}
\end{aligned}$$

which maximizes the overall social welfare (11a), given power balance constraints [(11b), (11c)]; thermal ampacity (11d) and voltage magnitude (11e) limits; and generation active/reactive limits (11f)/(11g). Variables listed right after the colon are duals of their respective constraints. In (11), note that, $(\tilde{p}^{\rm l},\tilde{q}^{\rm l},|\tilde{\mathbf{i}}_{ij}|,\tilde{\mathbf{v}}_{\rm L})$ are linear counterparts of their nonlinear functions, given as:

$$\tilde{\mathbf{v}}_{\rm L}=\hat{\mathbf{a}}+\mathbf{M}^{\mathbf{v}_{\rm L}}_{\mathbf{p}}\mathbf{p}_{\rm L}+\mathbf{M}^{\mathbf{v}_{\rm L}}_{\mathbf{q}_{\rm L}}\mathbf{q}_{\rm L}, \tag{12a}$$

$$|\tilde{\mathbf{i}}_{ij}|=\hat{\mathbf{b}}+\mathbf{M}^{|\mathbf{i}_{ij}|}_{\mathbf{p}}\mathbf{p}_{\rm L}+\mathbf{M}^{|\mathbf{i}_{ij}|}_{\mathbf{q}_{\rm L}}\mathbf{q}_{\rm L}, \tag{12b}$$

$$\tilde{p}^{\rm l}=\hat{c}+\mathbf{M}^{p^{\rm l}}_{\mathbf{p}_{\rm L}}\mathbf{p}_{\rm L}+\mathbf{M}^{p^{\rm l}}_{\mathbf{q}_{\rm L}}\mathbf{q}_{\rm L}, \tag{12c}$$

$$\tilde{q}^{\rm l}=\hat{d}+\mathbf{M}^{q^{\rm l}}_{\mathbf{p}}\mathbf{p}_{\rm L}+\mathbf{M}^{q^{\rm l}}_{\mathbf{q}_{\rm L}}\mathbf{q}_{\rm L}. \tag{12d}$$

See Section VI for derivations of (12).

Remark 5. With positive definite quadratic objective and affine constraints,the distribution grid ED problem (11) is a quadratic program (QP), which has a unique optimal $(\mathbf{p}^{*\rm dg},\mathbf{q}^{*\rm dg},p_0^*,q_0^*)$.

It can be seen that collecting all the information at a central location to solve (11) is a monumental task for the DSO. Also, DGs may not be comfortable sharing their private cost information with the DSO. This issue is addressed in the following proposed method.

III. Coordinated Market Mechanism

A. Method

1) At iteration $k=1$, the DSO initializes all duals of (11) and based on historic data, constructs sensitivity matrices in (12), to be passed to DGs
2) Each $i^{\rm th}$ DG updates[3] its generations as:

$$\begin{aligned}
(\mathbf{p}^{\rm dg})_i(k)=\Big[\Big(&\lambda_p(k)-(\mathbf{a}_{\mathbf{p}})_i-(\mathbf{M}^{p^{\rm l}}_{\mathbf{p}_{\rm L}})_i^\intercal\lambda_p(k)\\
&-(\mathbf{M}^{q^{\rm l}}_{\mathbf{p}_{\rm L}})_i^\intercal\lambda_q(k)+(\mathbf{M}^{\mathbf{v}_{\rm L}}_{\mathbf{p}_{\rm L}})_i^\intercal(\boldsymbol{\mu}^+_{\mathbf{v}_{\rm L}}(k)+\boldsymbol{\mu}^-_{\mathbf{v}_{\rm L}}(k))\\
&+(\mathbf{M}^{|\mathbf{i}_{ij}|}_{\mathbf{p}_{\rm L}})_i^\intercal\boldsymbol{\mu}^+_{|\mathbf{i}_{ij}|}(k)\Big)/(\mathbf{B}_{\mathbf{p}})_{i,i}\Big]^{\mathbf{p}^{\rm dg+}}_{\mathbf{p}^{\rm dg-}}
\end{aligned} \tag{13a}$$

$$\begin{aligned}
(\mathbf{q}^{\rm dg})_i(k)=\Big[\Big(&\lambda_q(k)-(\mathbf{a}_{\mathbf{q}})_i-(\mathbf{M}^{p^{\rm l}}_{\mathbf{q}_{\rm L}})_i^\intercal\lambda_p(k)\\
&-(\mathbf{M}^{q^{\rm l}}_{\mathbf{q}_{\rm L}})_i^\intercal\lambda_q(k)+(\mathbf{M}^{\mathbf{v}_{\rm L}}_{\mathbf{q}_{\rm L}})_i^\intercal(\boldsymbol{\mu}^+_{\mathbf{v}_{\rm L}}(k)-\boldsymbol{\mu}^-_{\mathbf{v}_{\rm L}}(k))\\
&+(\mathbf{M}^{|\mathbf{i}_{ij}|}_{\mathbf{q}_{\rm L}})_i^\intercal\boldsymbol{\mu}^+_{|\mathbf{i}_{ij}|}(k)\Big)/(\mathbf{B}_{\mathbf{q}})_{i,i}\Big]^{\mathbf{q}^{\rm dg+}}_{\mathbf{q}^{\rm dg-}}.
\end{aligned} \tag{13b}$$

[3]The operator $[(\cdot)]^a_b$ means that whenever $(\cdot)$ is outside the range $[a,b]$, it is simply projected at the closest boundary value, i.e., a or b.

3) The DSO collects the dispatch and updates: i) grid quantities,

$$\tilde{\mathbf{v}}_{\rm L}(k)=\hat{\mathbf{a}}+\mathbf{M}^{\mathbf{v}_{\rm L}}_{\mathbf{p}_{\rm L}}\mathbf{p}_{\rm L}(k)+\mathbf{M}^{\mathbf{v}_{\rm L}}_{\mathbf{q}_{\rm L}}\mathbf{q}_{\rm L}(k), \tag{14a}$$

$$|\tilde{\mathbf{i}}_{ij}|(k)=\hat{\mathbf{b}}+\mathbf{M}^{|\mathbf{i}_{ij}|}_{\mathbf{p}_{\rm L}}\mathbf{p}_{\rm L}(k)+\mathbf{M}^{|\mathbf{i}_{ij}|}_{\mathbf{q}_{\rm L}}\mathbf{q}_{\rm L}(k), \tag{14b}$$

$$\tilde{p}^{\rm l}(k)=\hat{c}+\mathbf{M}^{p^{\rm l}}_{\mathbf{p}_{\rm L}}\mathbf{p}_{\rm L}(k)+\mathbf{M}^{p^{\rm l}}_{\mathbf{q}_{\rm L}}\mathbf{q}_{\rm L}(k), \tag{14c}$$

$$\tilde{q}^{\rm l}(k)=\hat{d}+\mathbf{M}^{q^{\rm l}}_{\mathbf{p}_{\rm L}}\mathbf{p}_{\rm L}(k)+\mathbf{M}^{q^{\rm l}}_{\mathbf{q}_{\rm L}}\mathbf{q}_{\rm L}(k), \tag{14d}$$

ii) the required bulk interconnection power supply:

$$p_0(k)=(\lambda_p(k)-a_{p0})/\beta_{p0}, \tag{15a}$$

$$q_0(k)=(\lambda_q(k)-a_{q0})/\beta_{q0}, \tag{15b}$$

and iii) the duals:

$$\lambda_p(k+1)=\lambda_p(k)-\alpha_p\big(p_0(k)+\mathbf{1}_{\rm n}^\top\mathbf{p}_{\rm L}(k)+\tilde{p}^{\rm l}(k)\big), \tag{16a}$$

$$\lambda_q(k+1)=\lambda_q(k)-\alpha_q\big(q_0(k)+\mathbf{1}_{\rm n}^\top\mathbf{q}_{\rm L}(k)+\tilde{q}^{\rm l}(k)\big), \tag{16b}$$

$$\boldsymbol{\mu}^+_{|\mathbf{i}_{ij}|}(k+1)=\big[\boldsymbol{\mu}^+_{|\mathbf{i}_{ij}|}(k)-\beta\big(|\tilde{\mathbf{i}}_{ij}|(k)-|\mathbf{i}^+_{ij}|\big)\big]^\infty_0, \tag{16c}$$

$$\boldsymbol{\mu}^+_{\mathbf{v}_{\rm L}}(k+1)=\big[\boldsymbol{\mu}^+_{\mathbf{v}_{\rm L}}(k)-\gamma\big(\tilde{\mathbf{v}}_{\rm L}(k)-\mathbf{v}^+_{\rm L}\big)\big]^\infty_0, \tag{16d}$$

$$\boldsymbol{\mu}^-_{\mathbf{v}_{\rm L}}(k+1)=\big[\boldsymbol{\mu}^-_{\mathbf{v}_{\rm L}}(k)+\delta\big(\tilde{\mathbf{v}}_{\rm L}(k)-\mathbf{v}^-_{\rm L}\big)\big]^\infty_0. \tag{16e}$$

The updated duals are passed to DGs.
4) Step 2 and 3 are repeated for iteration $k=k+1$

The mechanism terminates when no improvements in duals and generation variables exists. The variables $(\alpha_p,\alpha_q,\beta,\gamma,\delta)$ are nonnegative constants serving as tuning parameter [20].

B. Interpretation

The iterative procedure of Section III-A has intuitive interpretation. The global active/reactive power balance dual λ_p/λ_q in (16a)/(16b) shows the mismatch between available generations and total demand, i.e., increases when generations are lower than the total demand. This increase in λ_p/λ_q in turn increases active/reactive power generation from the bulk interconnection using (15a)/(15b) and individual DGs using (13a)/(13b) to meet the total demand. Exactly similar explanation hold for the update of inequality constraint duals $(\boldsymbol{\mu}^+_{|\mathbf{i}_{ij}|},\boldsymbol{\mu}^+_{\mathbf{v}_{\rm L}},\boldsymbol{\mu}^-_{\mathbf{v}_{\rm L}})$. Note that the negative sign of $\mathbf{M}^{p^{\rm l}/q^{\rm l}}_{\mathbf{p}_{\rm L}}(\cdot)$ and $\mathbf{M}^{p^{\rm l}/q^{\rm l}}_{\mathbf{q}_{\rm L}}(\cdot)$ in (13) can be interpreted as an increase in the local generation decreases in the overall grid losses [17].

C. Optimality

Consider the following Lagrangian $\mathcal{L}$ of (11)

$$\begin{aligned}
\mathcal{L}=\;&w(\mathbf{p}^{\rm dg},\mathbf{q}^{\rm dg})-c(p_0)-c(q_0)-\lambda_p\big(p_0+\mathbf{1}_{\rm n}^\intercal\mathbf{p}_{\rm L}-\tilde{p}^{\rm l}\big)\\
&-\lambda_q\big(q_0+\mathbf{1}_{\rm n}^\intercal\mathbf{q}_{\rm L}-\tilde{q}^{\rm l}\big)-\boldsymbol{\mu}^{+\,\intercal}_{|\mathbf{i}_{ij}|}\big(|\tilde{\mathbf{i}}_{ij}|-|\mathbf{i}^+_{ij}|\big)-\boldsymbol{\mu}^{+\,\intercal}_{\mathbf{v}_{\rm L}}\big(\tilde{\mathbf{v}}_{\rm L}\\
&-\mathbf{v}^+_{\rm L}\big)+\boldsymbol{\mu}^{-\,\intercal}_{\mathbf{v}_{\rm L}}\big(\tilde{\mathbf{v}}_{\rm L}-\mathbf{v}^-_{\rm L}\big)-\boldsymbol{\mu}^{\rm dg+\,\intercal}_{\mathbf{p}}\big(\mathbf{p}^{\rm dg}-\mathbf{p}^{\rm dg+}\big)+\boldsymbol{\mu}^{\rm dg-\,\intercal}_{\mathbf{p}}\big(\mathbf{p}^{\rm dg}\\
&-\mathbf{p}^{\rm dg-}\big)-\boldsymbol{\mu}^{\rm dg+\,\intercal}_{\mathbf{q}}\big(\mathbf{q}^{\rm dg}-\mathbf{q}^{\rm dg+}\big)+\boldsymbol{\mu}^{\rm dg-\,\intercal}_{\mathbf{q}}\big(\mathbf{q}^{\rm dg}-\mathbf{q}^{\rm dg-}\big)
\end{aligned} \tag{17}$$

It can be seen that all generations and duals updates in Section III-A are in fact the first order derivatives of the above Lagrangian. To demonstrate this, consider the $i^{\rm th}$ DG update $(\mathbf{p}^{\rm dg})_i$ in (13a), where the terms inside $[\cdot]^{\mathbf{p}^{\rm dg+}}_{\mathbf{p}^{\rm dg-}}$ follows from $\frac{\partial\mathcal{L}}{\partial(\mathbf{p}^{\rm dg})_i}$. Similarly, λ_p update in (16a) follows from $\frac{\partial\mathcal{L}}{\partial\lambda_p}$. Moreover, note that all feasible space projections in Section III-A follows either the primal or dual space feasibility of (11). In this regards, it can be concluded that the proposed method is actually an iterative procedure for solving the KKT conditions of (11). Since for a strictly convex program the solution of its KKT conditions is necessary and sufficient for optimality, the proposed mechanism of this paper is also expected to obtain that optimal solution. Moreover, the iterative method of Section III-A can be considered as a form of gradient methods [20], where first order derivatives of the Lagrangian are used to project the solution to feasible spaces. For a strictly convex problems and a small enough positive $(\alpha_p, \alpha_q, \beta, \gamma, \delta)$, these methods are guaranteed to converge to a unique solution [20].

D. Practical Implications

The DSO handles all grid related quantities and has no information regarding its underlying DGs' costs' and energy requirements. Note that the sensitivity matrices of (12) have a very light computation burden. This is because for a fixed topology, ${\mathbf{Y}_{\mathrm{LL}}}^{-1}$ can be precomputed, allowing all sensitivity matrices to be found through simple arithmetics. Also, the voltage magnitude sensitivities are purely local, i.e., only local voltage measurements are needed to calculate them [15]. Moreover, for a radial grid, it can be seen that the line current magnitude sensitivities follow a recursive structure, i.e., for each branch, power flowing through a line can be recursively represented as sum of its respective children nodes (see [13, Sec. II-B] for more details).

Similarly, DGs can enjoy keeping their sensitive information private and only share their final generation with the DSO. Moreover, specific to DG location, part of the sensitivity matrix is multiplied with duals before being passed to individual DG. This also helps in decoding the actual grid condition (duals) from DGs, who may act strategically in the future, rather than rationally.

The convergence of the proposed method depends on the choice of tuning parameters. Exploring this issue, although an interesting topic, is out of scope for this paper and we postpone it as the future works. As long as the problem structure (convexity) of (11) is maintained, the framework proposed in this paper can be simply extended to include various market products (e.g. ancillary services), time horizons (day-ahead planning) and devices (battery energy storage systems). This also forms an interesting future works

IV. Simulation Setup and Results

The proposed method is tested on the 33-bus distribution system [21]. The bus connected to the bulk system is indexed as 0. Four DGs are included at bus 17, 21, 24 and 32 with active and reactive power dispatch within range [0, 0.2] MW and $[-0.1,\ 0.1]$ MVar. For all four DGs and bus 0, marginal cost of active and reactive power is set at 10 \$/MWh and 3 \$/MVarh. A small value of price sensitivity coefficient of 1.10^{-4} \$/MWh^2(MVarh^2) is chosen to keep all cost functions strictly convex. The tuning parameters are chosen as $(\alpha_p, \alpha_q, \beta, \gamma, \delta) = (0.01, 0.01, 0.1, 0.1, 0.1)$.

We consider two scenarios. Scenario 1 assumes no violation in grid and scenario 2 inflicts both voltage violation and line current congestion. To present the worst case, for both scenarios, all duals are initialized at zero (cold start). The proposed iterative method is terminated when constraints [(11b)–(11e)] are met within the tolerance of 1.10^{-6}. As expected, for both scenarios, identical results are obtained from the proposed method as compared to solving the problem (11) using an off-the-shelf solver such as GUROBI [22]. All simulations are carried out in MATLAB and terminated within 1 second, as throughout the iterative procedure only simple arithmetic terms are evaluated. All computations are performed on a computer with Intel i7 2.8 GhZ and 8 GB RAM. To visualize duals, we analyze the evolution the following compact price vector [11]:

$$\begin{aligned}
\mathbf{\Pi}_{\mathbf{p}_\mathrm{L}} &:= \lambda_p \mathbf{1}_\mathrm{n} - (\mathbf{M}^{p^\mathrm{l}}_{\mathbf{p}_\mathrm{L}})^\intercal \lambda_p - (\mathbf{M}^{q^\mathrm{l}}_{\mathbf{p}_\mathrm{L}})^\intercal \lambda_q + (\mathbf{M}^{|\mathbf{i}_{ij}|}_{\mathbf{p}_\mathrm{L}})^\intercal \boldsymbol{\mu}^+_{|\mathbf{i}_{ij}|} \\
&\quad + (\mathbf{M}^{\mathbf{v}_\mathrm{L}}_{\mathbf{p}_\mathrm{L}})^\intercal (\boldsymbol{\mu}^+_{\mathbf{v}_\mathrm{L}} - \boldsymbol{\mu}^-_{\mathbf{v}_\mathrm{L}}) \\
\mathbf{\Pi}_{\mathbf{q}_\mathrm{L}} &:= \lambda_q \mathbf{1}_\mathrm{n} - (\mathbf{M}^{p^\mathrm{l}}_{\mathbf{q}_\mathrm{L}})^\intercal \lambda_p - (\mathbf{M}^{q^\mathrm{l}}_{\mathbf{q}_\mathrm{L}})^\intercal \lambda_q + (\mathbf{M}^{|\mathbf{i}_{ij}|}_{\mathbf{q}_\mathrm{L}})^\intercal \boldsymbol{\mu}^+_{|\mathbf{i}_{ij}|} \\
&\quad + (\mathbf{M}^{\mathbf{v}_\mathrm{L}}_{\mathbf{q}_\mathrm{L}})^\intercal (\boldsymbol{\mu}^+_{\mathbf{v}_\mathrm{L}} - \boldsymbol{\mu}^-_{\mathbf{v}_\mathrm{L}}) \qquad (18)
\end{aligned}$$

For bus 0, we simply show the evolution of λ_p/λ_q which represent the marginal cost of supplying power to the grid (15). Section IV and 2 shows (18) for scenario 1 and 2, where solid lines represent duals from the proposed method and dashed lines the central solution of (11)

Scenario 1 – No Grid Violations: Since no constraint is binding in this scenario, (18) is only going to show the effect of losses. It can also be seen from Table I where, due to no grid constraints and competitive marginal offers, all DGs are dispatched to their maximum limits.

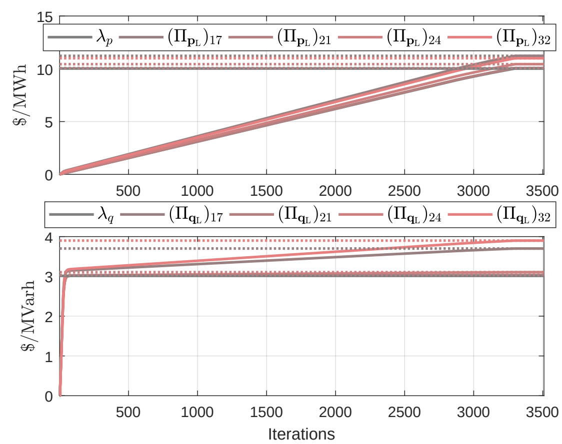

Fig. 1. Evolution of (18) for the scenario 1.

Scenario 2 – Grid Violations: The voltage at the bus 0 is fixed at 1.05 per unit which causes voltage binding at bus 21 and consequently limits DG 2 to dispatch less power Table I. Moreover, at the farthest branch (serving bus 32) line current is limited, which brings DG4 to a lower dispatch value as compared to scenario 1 (see Table I). Also, as compared to scenario 1, convergence for scenario-2 takes more iterations (see Section IV). This is because more duals are updated more frequently as more binding constraints exists.

Table I
Dispatch for scenario 1 (top) and scenario 2 (bottom)

Bus	0	17	21	24	32
MW	2.26	0.20	0.20	0.20	0.28
MVAR	1.21	0.10	0.10	0.10	0.10
MW	2.36	0.20	0.10	0.20	0.18
MVAR	1.30	0.10	0.00	0.10	0.10

V. Conclusion

We considered ED problem in distribution grids. In particular, we proposed an iterative market mechanism, which deployed approximated grid constraints and kept private information sensitive to respective entities. We highlighted the

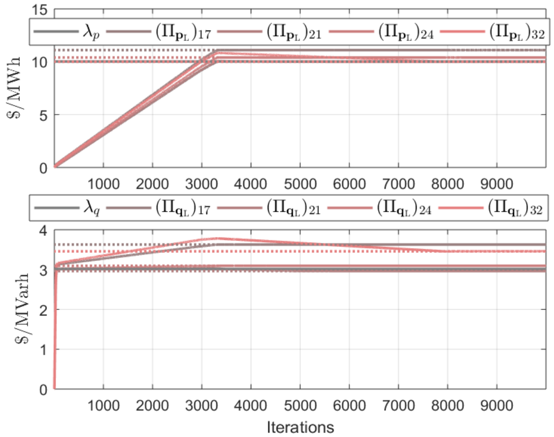

Fig. 2. Evolution of (18) for the scenario 2.

computational advantages of the proposed method as well as showed that it achieves identical solution to the central distribution grid ED problem.

VI. Appendix

Consider the first iteration from the satisfied fixed point *load-flow* equation (3), then the complex-valued voltage sensitivity with respect to n PQ grid injections is:

$$(\mathbf{M}^{\mathbf{u}_\mathrm{L}}_{\mathbf{p}_\mathrm{L}}, \mathbf{M}^{\mathbf{u}_\mathrm{L}}_{\mathbf{q}_\mathrm{L}}) := \Big(\frac{\partial \mathbf{u}_\mathrm{L}}{\partial \mathbf{p}_\mathrm{L}}, j\frac{\partial \mathbf{u}_\mathrm{L}}{\partial \mathbf{q}_\mathrm{L}}\Big) = \Big(\mathbf{Y}_{\mathrm{LL}}{}^{-1}\mathrm{diag}(\overline{\mathbf{u}}_\mathrm{L})^{-1}, -j\mathbf{Y}_{\mathrm{LL}}{}^{-1}\mathrm{diag}(\overline{\mathbf{u}}_\mathrm{L})^{-1}\Big).$$

From this, the linear complex voltage $\tilde{\mathbf{u}}_\mathrm{L}$ follows:

$$\tilde{\mathbf{u}}_\mathrm{L} = \mathbf{w} + \mathbf{M}^{\mathbf{u}_\mathrm{L}}_{\mathbf{p}_\mathrm{L}}\mathbf{p}_\mathrm{L} + \mathbf{M}^{\mathbf{u}_\mathrm{L}}_{\mathbf{q}_\mathrm{L}}\mathbf{q}_\mathrm{L} \tag{19}$$

a) Linear Voltage Magnitude (12a)*:* From [15], we follow a distributed method of obtaining linear voltage magnitude expression where $\hat{\mathbf{a}} = |\mathbf{w}|$, $\mathbf{M}^{\mathbf{v}_\mathrm{L}}_{\mathbf{p}_\mathrm{L}} := \Re(|\mathbf{W}|(\mathbf{W}^{-1}\mathbf{M}^{\mathbf{u}_\mathrm{L}}_{\mathbf{p}_\mathrm{L}}))$ and $\mathbf{M}^{\mathbf{v}_\mathrm{L}}_{\mathbf{q}_\mathrm{L}} := \Re(|\mathbf{W}|(\mathbf{W}^{-1}\mathbf{M}^{\mathbf{u}_\mathrm{L}}_{\mathbf{q}_\mathrm{L}}))$, with $\mathbf{W} := \mathrm{diag}(\mathbf{w})$.

b) Linear Line Current Magnitude (12b)*:* The complex linear current expression can be obtained as:

$$\tilde{\mathbf{i}}_{ij} = \mathbf{Y}^\mathrm{f}_0 u_0 + \mathbf{Y}^\mathrm{f}_\mathrm{L}\mathbf{w} + \mathbf{M}^{\mathbf{i}_{ij}}_{\mathbf{p}_\mathrm{L}}\mathbf{p}_\mathrm{L} + \mathbf{M}^{\mathbf{i}_{ij}}_{\mathbf{q}_\mathrm{L}}\mathbf{q}_\mathrm{L} \tag{20}$$

where $(\mathbf{M}^{\mathbf{i}_{ij}}_{\mathbf{p}_\mathrm{L}}, \mathbf{M}^{\mathbf{i}_{ij}}_{\mathbf{q}_\mathrm{L}}) := (\mathbf{Y}^\mathrm{f}_\mathrm{L}\mathbf{M}^{\mathbf{u}_\mathrm{L}}_{\mathbf{p}_\mathrm{L}}, \mathbf{Y}^\mathrm{f}_\mathrm{L}\mathbf{M}^{\mathbf{u}_\mathrm{L}}_{\mathbf{q}_\mathrm{L}}\mathbf{q}_\mathrm{L})$, allowing us to obtain the required parameters of (12b) as

$$\begin{aligned}
\mathbf{M}^{|\mathbf{i}_{ij}|}_{\mathbf{p}_\mathrm{L}} &:= \big(\mathrm{diag}(|\hat{\mathbf{i}}_{ij}|)\big)\Re\big(\mathrm{diag}(\bar{\hat{\mathbf{i}}}_{ij})\mathbf{M}^{\mathbf{i}_{ij}}_{\mathbf{p}_\mathrm{L}}\big),\\
\mathbf{M}^{|\mathbf{i}_{ij}|}_{\mathbf{q}_\mathrm{L}} &:= \big(\mathrm{diag}(|\hat{\mathbf{i}}_{ij}|)\big)\Re\big(\mathrm{diag}(\bar{\hat{\mathbf{i}}}_{ij})\mathbf{M}^{\mathbf{i}_{ij}}_{\mathbf{q}_\mathrm{L}}\big),\\
\hat{\mathbf{b}} &:= |\mathbf{Y}^\mathrm{f}_0 u_0 + \mathbf{Y}^\mathrm{f}_\mathrm{L}\mathbf{w}|
\end{aligned} \tag{21}$$

c) Linearized system losses (12c)*,* (12d)*:* From loss definition in (8), we have multiplication of a complex quantity with its own conjugate, this cancels out the imaginary components and gives the following form:

$$\frac{\partial s^\mathrm{l}}{\partial \mathbf{p}_\mathrm{L}} := 1^\intercal_\mathrm{m}\mathbf{Z}^\mathrm{l}\Bigg(2\Big(\mathrm{diag}\,\big(\Re(\hat{\mathbf{i}}_{ij})\big)\Re(\mathbf{M}^{\mathbf{i}_{ij}}_{\mathbf{p}_\mathrm{L}}) + \mathrm{diag}(\Im(\hat{\mathbf{i}}_{ij}))\Im(\mathbf{M}^{\mathbf{i}_{ij}}_{\mathbf{p}_\mathrm{L}})\Big)\Bigg),$$

$$\frac{\partial s^\mathrm{l}}{\partial \mathbf{q}_\mathrm{L}} := 1^\intercal_\mathrm{m}\mathbf{Z}^\mathrm{l}\Bigg(2\Big(\mathrm{diag}\,\big(\Re(\hat{\mathbf{i}}_{ij})\big)\Re(\mathbf{M}^{\mathbf{i}_{ij}}_{\mathbf{q}_\mathrm{L}}) + \mathrm{diag}(\Im(\hat{\mathbf{i}}_{ij}))\Im(\mathbf{M}^{\mathbf{i}_{ij}}_{\mathbf{q}_\mathrm{L}})\Big)\Bigg).$$

Since the term under big parenthesis is already real, the active and reactive power loss sensitivities are simply obtained as:

$$\mathbf{M}^{p^\mathrm{l}}_{\mathbf{p}_\mathrm{L}} := 1^\intercal_\mathrm{m}\mathbf{R}^\mathrm{l}\Bigg(2\Big(\mathrm{diag}\,\big(\Re(\hat{\mathbf{i}}_{ij})\big)\Re(\mathbf{M}^{\mathbf{i}_{ij}}_{\mathbf{p}_\mathrm{L}}) + \mathrm{diag}(\Im(\hat{\mathbf{i}}_{ij}))\Im(\mathbf{M}^{\mathbf{i}_{ij}}_{\mathbf{p}_\mathrm{L}})\Big)\Bigg),$$

$$\mathbf{M}^{q^\mathrm{l}}_{\mathbf{p}_\mathrm{L}} := 1^\intercal_\mathrm{m}\mathbf{X}^\mathrm{l}\Bigg(2\Big(\mathrm{diag}\,\big(\Re(\hat{\mathbf{i}}_{ij})\big)\Re(\mathbf{M}^{\mathbf{i}_{ij}}_{\mathbf{p}_\mathrm{L}}) + \mathrm{diag}(\Im(\hat{\mathbf{i}}_{ij}))\Im(\mathbf{M}^{\mathbf{i}_{ij}}_{\mathbf{p}_\mathrm{L}})\Big)\Bigg).$$

where $\mathbf{Z}^\mathrm{l} = \mathbf{R}^\mathrm{l} + j\mathbf{X}^\mathrm{l}$. Similar expressions hold for ($\mathbf{M}^{p^\mathrm{l}}_{\mathbf{q}_\mathrm{L}}$, $\mathbf{M}^{q^\mathrm{l}}_{\mathbf{q}_\mathrm{L}}$) and are not derived here in the interest of space. Note that for negligible shunt admittance, $\hat{d}$, $\hat{e}$ (no-load losses) can be assumed as zero [17].

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